BADJI MOKHTAR - ANNABA UNIVERSITY **2024-2025**
FACULTY OF ENGINEERING SCIENCES
Department of Science and Technology.
Prof. L. BECHIRI

Vibration and Waves
Final Examination
1 hour and 30 minutes.

Exercise 1 (4points)

A uniform rigid rod[1] of mass m, and length l is pinned[2] at point O as shown in figure 1. Its moment of inertia about its axis of rotation is given by: $J = \frac{ml^2}{3}$
Determine the natural angular frequency $\omega_n(\omega_0)$ of vibration of the rod for small angular rotation θ using the energy method.

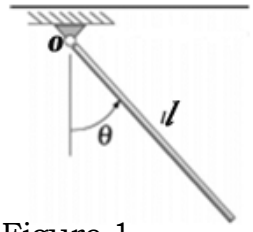


Figure 1

Exercise 2 (10points)

For a single degree of freedom spring–mass–damper system with $m = 3$kg , $k = 8$ N/m and $c = 8$Ns/m.
(a) Determine the equivalent spring constant k_{eq} (in N/m) for the spring configuration shown in the figure 2.
(b) Calculate the damping ratio ζ. Is this system overdamped, critically damped, or underdamped?

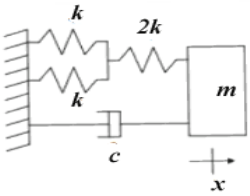
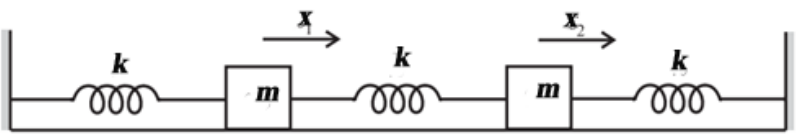


Figure 2

(c) Calculate the damped angular frequency ω_a.
(d) Find the solution $x_h(t)$ of the springs–mass–damper system as a function of an amplitude A and a phase φ.
(e) A motor is used to force the weight[4] to oscillate. After a long time, the transient[5] solution decays[6] so that the motion of the weight can be described by $x(t) = 0.02\, sin5t\ [m]$, where $x(t) = x_p(t)$ is the steady-state response (p = particular).
What is the driving force exerted[7] by the motor at time t = 8 s?

Exercise 3 (6points)

Identical springs, each having a stiffness k, are connected to two equal masses m as shown below. The masses are constrained[8] to move in one dimension on a frictionless horizontal surface.



1) Write the differential equations for each mass using Newton's second law (the fundamental equation of dynamics).
2) Calculate the natural angular frequencies of the system.
3) Determine the general expressions for $x_1(t)$ and $x_2(t)$.

Quelques mots traduits en français

[1]Tige, [2]Fixée, [3]Amortisseur, [4]Poids, [5]Régimes transitoires, [6]Diminue, [7]La force motrice exercée, [8]Contraintes

It is strictly forbidden for candidates to communicate with other candidates or external individuals during the exam, including the use of headphones. Additionally, any exchange of materials such as pens, rulers, calculators, and erasers is strictly forbidden.

L'énoncé en français se trouve au verso

Exercice 1 (4points)

Une tige rigide uniforme de masse m et de longueur l est fixée en un point O comme indiqué sur la figure 1. Son moment d'inertie par rapport à l'axe de rotation est donné par : $J = \frac{ml^2}{3}$

Déterminez la fréquence angulaire naturelle $\omega_n(\omega_0)$ de vibration de la tige pour de faibles angles de rotation θ en utilisant la méthode énergétique.

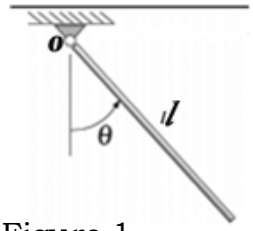


Figure 1

Exercice 2 (10points)

Pour un système masse-ressort-amortisseur à un degré de liberté avec $m = 3\text{kg}$, $k = 8\ \text{N/m}$ and $c = 8\text{Ns/m}$.

(a) Déterminez la constante de raideur équivalente k_{eq} (en N/m) pour la configuration de ressorts montrée sur la figure 2.

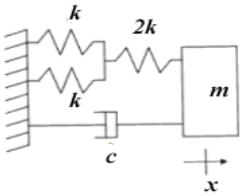


Figure 2

(b) Calculez le coefficient d'amortissement ζ. Ce système est-il sur-amorti, amorti critique, ou sous-amorti ?

(c) Calculez la fréquence angulaire amortie ω_a

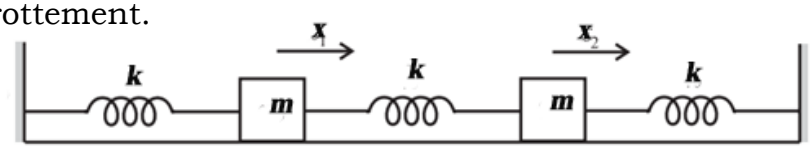
(d) Trouvez la solution $(x_h(t))$ du système masse-ressort-amortisseur en fonction d'une amplitude A et d'une phase φ.

(e) Un moteur est utilisé pour forcer le poids à osciller. Après un long moment, la solution transitoire s'annule de sorte que le mouvement du poids peut être décrit par $x(t) = 0.02\, sin5t\ [m]$, où $x(t) = x_p(t)$ est la réponse en régime permanent (p = particulière).

Quelle est la force motrice exercée par le moteur à t = 8 s ?

Exercice 3 (6points)

Des ressorts identiques, chacun ayant une constante de force (raideur) k, sont connectés à deux masses égales m comme indiqué ci-dessous. Les masses sont contraintes à se déplacer dans une dimension sur une surface horizontale sans frottement.



1)Déterminez les équations différentielles pour chaque masse en utilisant la deuxième loi de Newton (la relation fondamentale de la dynamique).

2)Calculez les pulsations propres (fréquences angulaires) du système.

3)Déterminez les expressions générales pour $x_1(t)$ and $x_2(t)$.

Instructions pour les candidats

Il est strictement interdit aux candidats de communiquer avec d'autres candidats ou des personnes extérieures pendant l'examen, y compris l'utilisation de casques. De plus, tout échange de matériel tel que stylos, règles, calculatrices et gommes est strictement interdit.

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Vibration and Waves
Solutions of Final Examination

Solution 1: (4points)

Using the notation and coordinates of the figure. If the mass of the pendulum rod is m, and it is concentrated at the center of mass, the energies become:

Kinetic energy:

$$T = T_{rod} = \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}(\frac{ml^2}{3})\dot{\theta}^2 = \left(\frac{1}{6}m\right)l^2\dot{\theta}^2 \qquad \textbf{(0.5)}$$

Potential energy: $V = V_{rod} = mgh$ with $h = \frac{1}{2}(l - lcos\theta)$ **(0.25)**

$$V_{rod} = \frac{1}{2}(l - lcos\theta)mg = \frac{l}{2}(1 - cos\theta)(mg) \qquad \textbf{(0.5)}$$

Energy conservation requires: $T + V = constant$:

$$\left(\frac{1}{6}m\right)l^2\dot{\theta}^2 + \frac{l}{2}(1 - cos\theta)(mg) = C \qquad \textbf{(0.5)}$$

Differentiating with respect to time, we obtain:

$$\frac{d(E)}{dt} = \frac{d(T+V)}{dt} = 0 \qquad \textbf{(0.25)}$$

(Applying the Chain Rule to find the derivative of a composition of functions $\frac{d\dot{\theta}^2}{dt} = \frac{d\dot{\theta}^2}{d\theta}\frac{d\theta}{dt} = 2\ddot{\theta}\dot{\theta}$)

$$\left(\frac{1}{3}m\right)\dot{\theta}\ddot{\theta} + \frac{l}{2}(sin\theta)(mg)\dot{\theta} = 0 \qquad \textbf{(0.5)}$$

$$\Longrightarrow \left(\frac{1}{3}m\right)l\ddot{\theta} + \frac{1}{2}(mg)sin\theta = 0 \qquad \textbf{(0.5)}$$

Reorganizing and approaching the expression using the small angle approximation $sin\theta \approx \theta$, we obtain:

$$\ddot{\theta} + \left(\frac{\frac{m}{2}}{\frac{1}{3}m}\frac{g}{l}\right)\theta = 0 \Longrightarrow \ddot{\theta} + \frac{3}{2}\frac{g}{l}\theta \Longrightarrow \ddot{\theta} + \omega_n^2\theta = 0. \qquad \textbf{(0.5)}$$

This is the differential equation for a simple harmonic oscillator with:

$$\omega_n = \sqrt{\frac{3}{2}\frac{g}{l}}\,rad/s \qquad \textbf{(0.5)}$$

<u>Solution 2 :</u> (10points)

a)Equivalent Spring Constant Calculation:

*Parallel Combination: Two springs with spring constant k in parallel:

$$k_{parallel} = k + k = 2k \qquad \textbf{(0.5)}$$

*Series Combination: The parallel combination (with spring constant $2k$) in series with another spring with spring constant $2k$:

$$\frac{1}{k_{eq}} = \frac{1}{k_{parallal}} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{2}{2k} \qquad \textbf{(0.5)}$$

The equivalent spring constant : $k_{eq} = k = 8\ N/m$ **(0.5)**

(b) Calculate the damping ratio. Using the standard formulas for angular frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8}{3}} \approx 1.63 \text{ rad/s} \qquad \textbf{(0.5)} + \textbf{(0.25)}$$

The damping ratio ζ is given by:

$$\zeta = \frac{c}{2m\omega_n} = \frac{8}{2(3)(1.63)} = 0.82 \ (or\ \zeta = \frac{c}{2\sqrt{mk_{eq}}} = 0.82) \qquad \textbf{(0.5)} + \textbf{(0.25)}$$

$$\zeta = 0.82 < 1$$

Since ζ is less than 1, the system is underdamped and will oscillate. **(0.5)**

c) The Damped natural frequency is therefore: $\zeta = \frac{c}{2m\omega_n} = \frac{\lambda}{\omega_n}$,

$$\omega_a = \sqrt{\omega_n^2 - \lambda^2}, \qquad \omega_n\zeta = \lambda \qquad \textbf{(0.25)+ (0.25)}$$

$$\omega_a = \omega_n\sqrt{1-\zeta^2} = 1.63 * \sqrt{1-(0.82)^2} \approx 0.93 \text{ rad/s}. \qquad \textbf{(0.5)} + \textbf{(0.5)}$$

d) The response of springs–mass–damper system is:

$$x(t) = Ae^{-\lambda t}\cos(\omega_a t + \varphi) \qquad \textbf{(0.5)}$$

$$\lambda = \frac{c}{2m},\ \lambda = \frac{8}{2x3} = 1.33, \qquad \textbf{(0.5)} + \textbf{(0.5)}$$

$$x(t) = Ae^{-1.33t}\cos(0.93t + \varphi) \qquad \textbf{(0.5)}$$

e)The amplitude of the driving force that the motor exerts can be determined from the steady-state solution,

$$X_0 = |\overline{X}| = \frac{F_0}{\sqrt{(k - m\omega_{ex}^2)^2 + (c\omega_{ex})^2}} \ \left(X_0 = |\overline{X}| = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega_{ex}^2)^2 + (2\lambda\omega_{ex})^2}} \right) \qquad \textbf{(0.5)}$$

Here $X_0 = 0{,}02\ [m]$ and $\omega_{ex} = 5\ [rad/s]$.

The amplitude of the driving force is,

$$F_0 = 1.56\ [N]$$ **(0.5)**

The phase between the force and the steady-state response is,

$$\phi_d = -\tan^{-1}\left(\frac{c\omega_{ex}}{k - m\omega_{ex}^2}\right) \quad (with\ \omega_n^2 = \frac{k}{m}\ )$$ **(0.5)**

$$(\arg(-z) = \arg(z) + \pi)$$

$$(\phi_d = \pi + tang^{-1}x)$$

$$\phi_d = \pi - 0{,}5369 = 2.60\ [rad].$$ **(0.5)**

The driving force leads the steady-state response by an angle ϕ_d,

$$F_{ex}(t) = F_0 sin(\omega_{ex}t + \phi_d)$$ **(0.5)**

$$F_{ex}(t) = 1.56 sin(5t + 2.60)$$ **(0.25)**

$$F_{ex}(t) = 1.56\, sin(5 * 8 + 2.60) = 1.56\, sin\ (42.60)$$

At time t = 8 s, , the force is, $F_{ex} = -1.53\ [N]$ **(0.25)**

Solution 3: (6points)

(a) The equations of motion can be derived using Newton's second law (the fundamental equation of dynamics). The forces acting on the two masses are the external forces $F_1 = -kx_1$ and $F_2 = -kx_2$, and the intermass forces $F_{21} = k(x_2 - x_1) = -F_{12}$. Consequently, the equations of motion are

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) \qquad \textbf{(0,5)}$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) \qquad \textbf{(0,5)}$$

Rearranging these equations, we get:

$$\begin{cases} m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \end{cases} \qquad (1) \qquad \textbf{(0,25)}$$

(0,25)

b) Natural Frequencies

To find the natural frequencies, we assume a solution of the form:

$$x_1(t) = A\cos(\omega t + \varphi) \qquad \textbf{(0,5)}$$

And

$$x_2(t) = B\cos(\omega t + \varphi) \qquad \textbf{(0,5)}$$

where ω is the angular frequency and φ is the initial phase.

Then

$$\ddot{x}_1 = -\omega^2 x_1$$

And

$$\ddot{x}_2 = -\omega^2 x_2$$

Substituting these into the equations of motion (1) and simplifying, we get a system of linear equations:

$$\begin{cases} (2k - m\omega^2)A - kB = 0 \\ -kA + (2k - m\omega^2)B = 0 \end{cases} \qquad \binom{a}{b} \qquad \binom{\textbf{0,25}}{\textbf{0,25}}$$

This can be written in matrix form as:

$$\begin{vmatrix} [(2k - m\omega^2)] & -k \\ -k & [(2k - m\omega^2)] \end{vmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

The system is homogeneous. There are non-trivial solutions only if the determinant $\mathfrak{D}$ of the coefficient matrix is zero. The coefficients are polynomials in ω. The permissible values are the solutions of $\mathfrak{D} = 0$.

$$\mathfrak{D} = \begin{vmatrix} [(2k - m\omega^2)] & -k \\ -k & [(2k - m\omega^2)] \end{vmatrix} = O \qquad \textbf{(0,25)}$$

The natural frequencies of the system are the roots of the equation:

$$\mathfrak{D} = \begin{vmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{vmatrix} = (2k-m\omega^2)^2 - k^2 = (3k-m\omega^2)(k-m\omega^2) = 0 \qquad \textbf{(0,25)}$$

Thus, the natural frequencies are:

$$\Rightarrow \omega_1 = \sqrt{\frac{3k}{m}} \quad rad/s \qquad \textbf{(0.5)}$$

$$and \qquad \omega_2 = \sqrt{\frac{k}{m}} \quad rad/s \quad . \qquad \textbf{(0.5)}$$

c)- For mode 1, $\omega = \omega_1 = \sqrt{\frac{3k}{m}} \quad rad/s$

$$x_1(t) = A_1 cos(\omega_1 t + \varphi_1) \; and \; x_2(t) = B_1 cos(\omega_1 t + \varphi_1)$$

And for mode 2, $\omega = \omega_2 = \sqrt{\frac{k}{m}} \quad rad/s$

$$x_1(t) = A_2 cos(\omega_2 t + \varphi_1) \; and \; x_2(t) = B_2 cos(\omega_2 t + \varphi_1)$$

The homogeneous solutions are then:

$$x_1(t) = x_{1mode1} + x_{1mode2} = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \qquad \textbf{(0,25)}$$

$$x_2(t) = x_{2mode1} + x_{2mode2} = B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2) \qquad \textbf{(0,25)}$$

Where A_1, A_2, B_1, B_2, φ_1, and φ_2 are constants determined by the initial conditions of the system. To reduce the number of unknowns in the previous solutions from six to four, we use the ratio of the amplitudes. Starting from equation (a) or (b), we deduce:

$$\frac{A}{B} = \frac{k}{2k - m\omega^2}$$

Mode 1, $\omega = \omega_1 = \sqrt{\frac{3k}{m}} \quad rad/s, \frac{A_1}{B_1} = \left(\frac{A}{B}\right)_{mode1} = \frac{k}{2k-m\omega_1^2} = -1, \; A_1 = -B_1$ **(0,25)**

Mode 2, $\omega = \omega_2 = \sqrt{\frac{k}{m}} \quad rad/s, \frac{A_2}{B_2} = \left(\frac{A}{B}\right)_{mode2} = \frac{k}{2k-m\omega_2^2} = 1, \quad A_2 = B_2$ **(0,25)**

The expressions $x_1(t)$ and $x_2(t)$ can be written as:

$$x_1(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \qquad \textbf{(0,25)}$$

$$x_2(t) = -A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \qquad \textbf{(0,25)}$$